

# Estimating quantiles

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March 28, 2025

The  $p$ th quantile is defined as the value where the estimated cumulative distribution function is equal to  $p$ . As with quantiles in unweighted data, this definition only pins down the quantile to an interval between two observations, and a rule is needed to interpolate. As the help for the base R function `quantile` explains, even before considering sampling weights there are many possible rules.

Rules in the `svyquantile()` function can be divided into three classes

- Discrete rules, following types 1 to 3 in `quantile`
- Continuous rules, following types 4 to 9 in `quantile`
- A rule proposed by Shah & Vaish (2006) and used in some versions of SUDAAN

## Discrete rules

These are based on the discrete empirical CDF that puts weight proportional to the weight  $w_k$  on values  $x_k$ .

$$\hat{F}(x) = \frac{\sum_i \{x_i \leq x\} w_i}{\sum_i w_i}$$

**The mathematical inverse** The mathematical inverse  $\hat{F}^{-1}(p)$  of the CDF is the smallest  $x$  such that  $F(x) \geq p$ . This is rule `hf1` and `math` and in equally-weighted data gives the same answer as `type=1` in `quantile`

**The primary-school median** The school definition of the median for an even number of observations is the average of the middle two observations. We extend this to say that the  $p$ th quantile is  $q_{\text{low}} = \hat{F}^{-1}(p)$  if  $\hat{F}(q_{\text{low}}) = p$  and otherwise is the the average of  $\hat{F}^{-1}(p)$  and the next higher observation. This is `school` and `hf2` and is the same as `type=2` in `quantile`.

**Nearest even order statistic** The  $p$ th quantile is whichever of  $\hat{F}^{-1}(p)$  and the next higher observation is at an even-numbered position when the distinct data values are sorted. This is `hf3` and is the same as `type=3` in `quantile`.

## Continuous rules

These construct the empirical CDF as a piecewise-linear function and read off the quantile. They differ in the choice of points to interpolate. Hyndman & Fan describe these as interpolating the points  $(p_k, x_k)$  where  $p_k$  is defined in terms of  $k$  and  $n$ . For weighted use they have been redefined in terms of the cumulative weights  $C_k = \sum_{i \leq k} w_i$ , the total weight  $C_n = \sum w_i$ , and the weight  $w_k$  on the  $k$ th observation.

qrule	Hyndman & Fan	Weighted
hf4	$p_k = k/n$	$p_k = C_k/C_n$
hf5	$p_k = (k - 0.5)/n$	$p_k = (C_k - w_k)/C_n$
hf6	$p_k = k/(n + 1)$	$p_k = C_k/(C_n + w_n)$
hf7	$p_k = (k - 1)/(n - 1)$	$p_k = C_{k-1}/C_{n-1}$
hf8	$p_k = (k - 1/3)/(n + 2/3)$	$p_k = (C_k - w_k/3)/(C_n + w_n/3)$
hf9	$p_k = (k - 3/8)/(n + 1/4)$	$p_k = (C_k - 3w_k/8)/(C_n + w_n/4)$

## Shah & Vaish

This rule is related to hf6, but it is discrete and more complicated. First, define  $w_i^* = w_i n / C_n$ , so that  $w_i^*$  sum to the sample size rather than the population size, and  $C^*k$  as partial sums of  $w_k^*$ . Now define the estimated CDF by

$$\hat{F}(x_k) = \frac{1}{n+1} (C_k^* + 1/2 - w_k/2)$$

and take  $\hat{F}^{-1}(p)$  as the  $p$ th quantile.

## Other options

It would be possible to redefine all the continuous estimators in terms of  $w^*$ , so that type 8, for example, would use

$$p_k = (C_k^* - 1/3)/(C_n^* + 2/3)$$

Or a compromise, eg using  $w_k^*$  in the numerator and numbers in the denominator, such as

$$p_k = (C_k^* - w_k^*/3)/(C_n^* + 2/3).$$

Comparing these would be ~~a worthwhile~~... an interesting... a research question for simulation.