Spearman's Rho for the AMH Copula: a Beautiful Formula

Martin Mächler
ETH Zurich
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Abstract

We derive a beautiful series expansion for Spearman's rho, $\rho(\theta)$ of the Ali-Mikhail-Haq (AMH) copula with paramater θ which is also called α or θ. Further, via experiments we determine the cutoffs to be used for practically fast and accurate computation of $\rho(\theta)$ for all $\theta \in [-1, 1]$.

Keywords: Archimedean copulas, Spearman's rho.

1. Introduction

A *copula* is a multivariate distribution function with standard uniform univariate margins. Standard references for an introduction are Joe (1997) or Nelsen (2007).

Sklar (1959) shows that for any multivariate distribution function H with margins F_j, $j \in \{1, \dots, d\}$, there exists a copula C such that

$$H(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)), \ \boldsymbol{x} \in \mathbb{R}^d. \tag{1}$$

Conversely, given a copula C and arbitrary univariate distribution functions F_j, $j \in \{1, \dots, d\}$, H defined by (1) is a distribution function with marginals F_j, $j \in \{1, \dots, d\}$.

2. Archimedean copulas

An *Archimedean generator*, or simply *generator*, is a continuous, decreasing function $\psi : [0, \infty] \to [0, 1]$ which satisfies $\psi(0) = 1$, $\psi(\infty) := \lim_{t\to\infty} \psi(t) = 0$, and which is strictly decreasing on $[0, \inf\{t : \psi(t) = 0\}]$. A d-dimensional copula is called *Archimedean* if it is of the form

$$C(\boldsymbol{u}; \psi) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)), \ \boldsymbol{u} \in [0, 1]^d, \tag{2}$$

for some generator ψ with inverse $\psi^{-1} : [0, 1] \to [0, \infty]$, where $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$. A necessary and sufficient condition for an Archimedean generator ψ to generate a proper copula in all dimensions d is that ψ is *completely monotone*, i.e., $(-1)^k \psi^{(k)}(t) \geq 0$ for all $t \in (0, \infty)$ and $k \in \mathbb{N}_0$. See Hofert and Maechler (2011) and its references, for considerably more details.

2.1. The Ali-Mikhail-Haq (AMH) copulas

An Ali-Mikhail-Haq (AMH) copula with parameter θ, $\theta \in [-1, 1)$ (where the right boundary, $\theta = 1$ can sometimes be considered valid) has generator

$$\psi_{\text{AMH}}(t, \theta) = \frac{1 - \theta}{\exp(t) - \theta}. \tag{3}$$

For, $\theta = 0$, clearly $\psi(t) = \exp(-t)$, corresponds to independence. Both "rank based" association measures or correlations, Kendall's τ and Spearman's ρ, are montone in θ, and hence have the same sign as θ.

Kendall's tau is equal to

$$\tau_\theta = 1 - \frac{2((1-\theta)^2 \log(1-\theta) + \theta)}{3\theta^2}, \tag{4}$$

for $\theta \in [0, 1)$, τ is in $[0, \frac{1}{3})$. The formula (4) needs care when θ is close to zero, and we provide `tauAMH()` in the **copula** package, using a Taylor series for small $|\theta|$, see `help(tauAMH)`.

3. Spearman's Rho (ρ) for AMH

3.1. The beautiful formula

Nelsen (2007, ex. 5.10, p. 172) provides the following formula for Spearman's ρ for the AMH copula,

$$\rho(\theta) = \frac{12(1+\theta)}{\theta^2} \cdot \text{dilog}(1-\theta) - \frac{24(1-\theta)}{\theta^2} \cdot \log(1-\theta) - \frac{3(\theta+12)}{\theta}, \tag{5}$$

where his "dilogarithm" $\text{dilog}(x) = \text{Li}_2(1-x) =$ `polylog`$(1-x, 2)$, and $\text{Li}_2(x)$ is the usual definition of the dilogarithm (also called "Spence's function"),

$$\text{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u}\,\mathrm{d}u = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad z \in \mathbb{C} \setminus [1, \infty), \tag{6}$$

where the infinite sum is only applicable for $|z| < 1$.

With the boundaries for $\theta \in \{-1, 1\}$, this leads to a range of ρ in the interval $[33 - 48\log 2, 4\pi^2 - 39]$ or approximately $[-0.2711, 0.4784]$.

It is clear that formula (5) cannot be used for $\theta = 0$ and further inspection reveals that it also heavily suffers from cancellation for $|\theta| \ll 1$.

In order to compute ρ accurately for all values of θ, we look at the Taylor series of the respective terms in (5) and will find a beautiful infinite series formula for $\rho(\theta)$.

$$\begin{aligned}
\rho(\theta) &= \frac{12(1+\theta)}{\theta^2} \cdot \text{Li}_2(\theta) - \frac{24(1-\theta)}{\theta^2} \cdot \log(1-\theta) - \frac{3(\theta+12)}{\theta} \\
&= 3/\theta \cdot (4(1+\theta)/\theta \cdot \text{Li}_2(\theta) - 8(1-\theta)/\theta \cdot \log(1-\theta) - (\theta+12)) \\
&= \frac{3}{\theta} \cdot r(\theta), \quad \text{where} \qquad (7) \\
r(\theta) &:= 4(1+\frac{1}{\theta}) \cdot \text{Li}_2(\theta) - 8(\frac{1}{\theta} - 1) \cdot \log(1-\theta) - (\theta+12). \qquad (8)
\end{aligned}$$

Now, we plug in the Taylor series of both $\mathrm{Li}_2(\theta) = \sum_{k=1}^{\infty} \frac{\theta^k}{k^2}$, hence

$$r_1(\theta) := (1 + \frac{1}{\theta}) \cdot \mathrm{Li}_2(\theta) = \mathrm{Li}_2(\theta) + \frac{1}{\theta} \cdot \mathrm{Li}_2(\theta) = \sum_{k=1}^{\infty} \frac{\theta^k}{k^2} + \sum_{k=1}^{\infty} \frac{\theta^{k-1}}{k^2} =$$
$$= 1 + \sum_{k=1}^{\infty} \frac{k^2 + (k+1)^2}{k^2(k+1)^2} \theta^k, \tag{9}$$

and $\log(1-\theta) = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \ldots = \sum_{k=1}^{\infty} \frac{\theta^k}{k}$, hence

$$r_2(\theta) := (1 - \frac{1}{\theta}) \log(1-\theta) = \sum_{k=1}^{\infty} \frac{\theta^k}{k} - \sum_{k=1}^{\infty} \frac{\theta^{k-1}}{k} = -1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k(k+1)}. \tag{10}$$

Consequently, first from (8), then plugging in (9) and (10),

$$\begin{aligned}
r(\theta) &= 4r_1(\theta) - 8r_2(\theta) - (12 + \theta) = \\
&= (4 \cdot 1 - 8(-1) - 12) + (4 \cdot \frac{5}{4} - 8 \cdot \frac{1}{2} - 1)\theta + \sum_{k=2}^{\infty} \Big(\frac{4(k^2 + (k+1)^2)}{k^2(k+1)^2} - \frac{8}{k(k+1)} \Big) \theta^k = \\
&= 0 + 0 \cdot \theta + \sum_{k=2}^{\infty} \frac{4(k^2 + (k+1)^2) - 8k(k+1)}{k^2(k+1)^2} \theta^k = \\
&= \sum_{k=2}^{\infty} \frac{4(2k^2 + 2k + 1)^2 - 8k(k+1)}{k^2(k+1)^2} \theta^k = \\
&= \sum_{k=2}^{\infty} \frac{4}{k^2(k+1)^2} \theta^k = \sum_{k=2}^{\infty} \frac{\theta^k}{\binom{k+1}{2}^2},
\end{aligned} \tag{11}$$

a beautiful formula with reciprocal binomial coefficients, and finally, as $\rho(\theta) = \frac{3}{\theta} \cdot r(\theta)$ (7) from the above,

$$\rho(\theta) = \sum_{k=1}^{\infty} \frac{3}{\binom{k+2}{2}^2} \cdot \theta^k = \frac{\theta}{3} + \frac{\theta^2}{12} + \frac{3\theta^3}{100} + \frac{\theta^4}{75} + \ldots \tag{12}$$

the "beautiful formula" for Spearman's ρ of an AMH copula with parameter θ. Compare this compact formula

$$\boxed{\rho(\theta) = \sum_{k=1}^{\infty} \frac{3\theta^k}{\binom{k+2}{2}^2}}$$

with the original three term formula (5) which involves dilog() and log(), to understand why I call it *beautiful*. Note further that the "beautiful formula" clearly shows the approximate linearity of $\rho(\theta)$ for small $|\theta|$. Note that the first few coefficients a_k in $\rho(\theta) = \sum_{k=1}^{\infty} a_k \theta^k$ are

```
> require(sfsmisc) #--> mat2tex(), mult.fig(), eaxis()
> k <- 1:9; ak <- MASS::fractions(12/((k+1)*(k+2))^2)
> rbind(k = k, `$a_k$` = as.character(ak))
```

k	1	2	3	4	5	6	7	8	9
a_k	1/3	1/12	3/100	1/75	1/147	3/784	1/432	1/675	3/3025

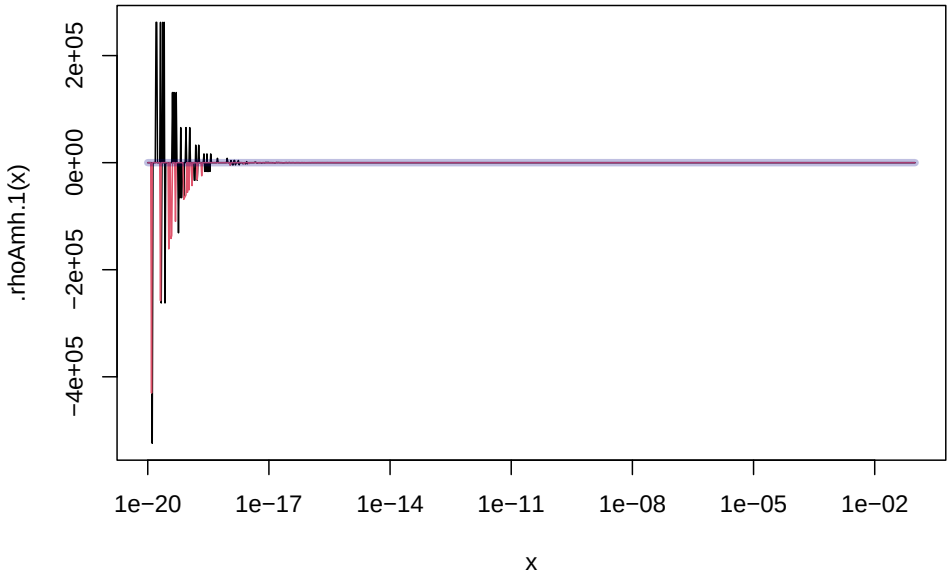
3.2. Accurate and efficient R implementation of ρ_{AMH}

In the following R code, we use `a` as short form for the copula parameter θ (which is also called α in the literature):

```
> ##' Version 1:  Direct formula from Nelsen:
> .rhoAmh.1 <- function(a) {
     Li2 <- gsl::dilog(a)
     12 * (1 + a) / a^2 * Li2 - 24 * (1 - a) / a^2 * log1p(- a) - 3 * (a + 12) / a
   }
> .rhoAmh.1b <- function(a) {
     Li2 <- gsl::dilog(a)
     ## factored out 3/a from version 1:
     3/a * (4 * (1 + a) / a * Li2 - 8 * (1 - a) / a * log1p(- a) - (a + 12))
   }
> ##' Version 2:
> .rhoAmh.2 <- function(a, e.sml = 1e-11) {
     stopifnot(length(a) <= 1)
     if(abs(a) < e.sml) { ## if |a| << 1, do better than the direct formula:
         a*(1/3 + a*(1/12 + a*(3/100 + a/75)))
     } else { ## regular a
         Li2 <- gsl::dilog(a)
         3/a * (4 * (1 + 1/a) * Li2 - 8 * (1/a - 1) * log1p(- a) - (a + 12))
     }
   }
> ##' Series version with N terms:
> rhoAmh.T <- function(a, N) {
       stopifnot(length(N) == 1, N == as.integer(N), N >= 1)
       if(N <= 4)
           switch(N,
                  a/3,
                  a/3*(1 + a/4),
                  a*(1/3 + a*(1/12 + a* 3/100)),
                  a*(1/3 + a*(1/12 + a*(3/100 + a/75))))
       else { ## N >= 5
           n <- N:1 #--> sum smallest to largest
           if(is(a, "mpfr")) ## so all computations work in high precision
               n <- mpfr(n, precBits=max(.getPrec(a)))
           cf <- ## 3/choose(n+2, 2)^2
               3/((n+1)*(n+2)/2)^2
           a2n <- outer(n,a, function(x,y) y^x) ## a2n[i,j] := a[j] ^ n[i]
           colSums(cf * a2n)
       }
   }
```

Now, the first graphical exploration, notably of the original Nelsen formula, `.rhoAmh.1()` and its variant very slight improvement `.rhoAmh.1b()`

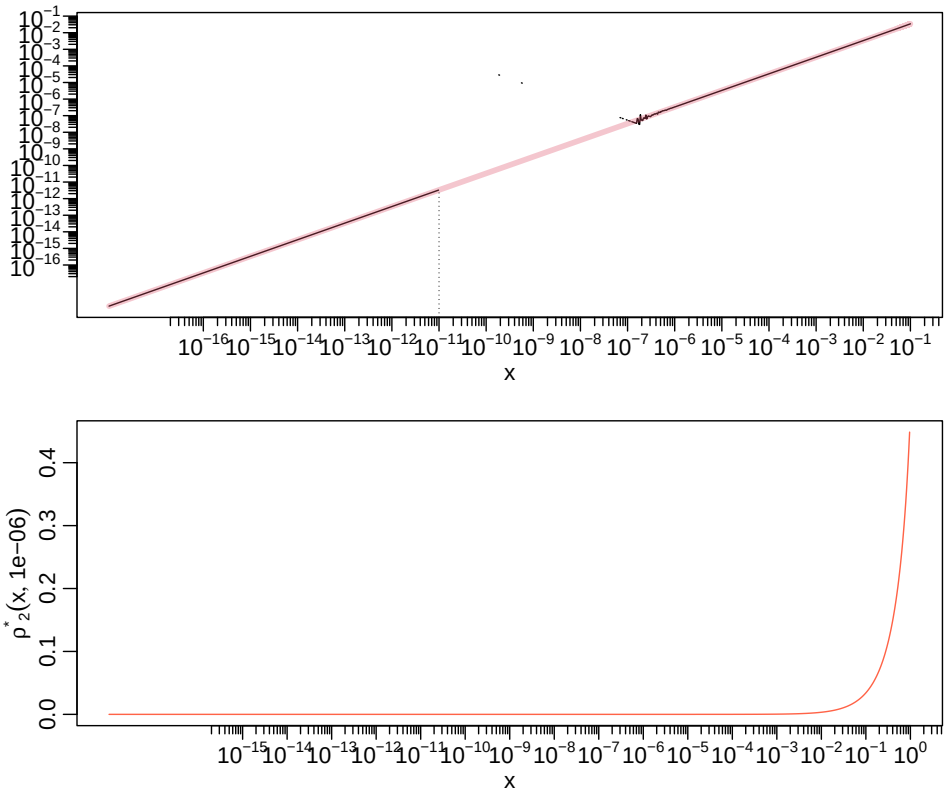
```
> r1  <- curve( .rhoAmh.1 (x),  1e-20, .1, log="x", n=1025)
> r1b <- curve( .rhoAmh.1b(x), n=1025, add=TRUE, col=2)
> r2 <-  curve( Vectorize(.rhoAmh.2)(x), n=1025, add=TRUE,
                col=adjustcolor("blue4",1/4), lwd = 5)
> tab <- cbind(as.data.frame(r1), y.b = r1b$y, y2 = r2$y)
```



expose the big problems (y-values between -400'000 and 200'000 where $|\rho()| < 1$ is known!). Investingating `tab` shows that `1b` is very slightly better than `1`, but looking closer, e.g. also with `curve(.rhoAmh.1(x), 1e-20, .1, log="x", n=1025, ylim=c(-1,1)*.1)`, shows that Nelsen's direct formula is really unusable for $|\theta| < 10^{-11}$ approximately.

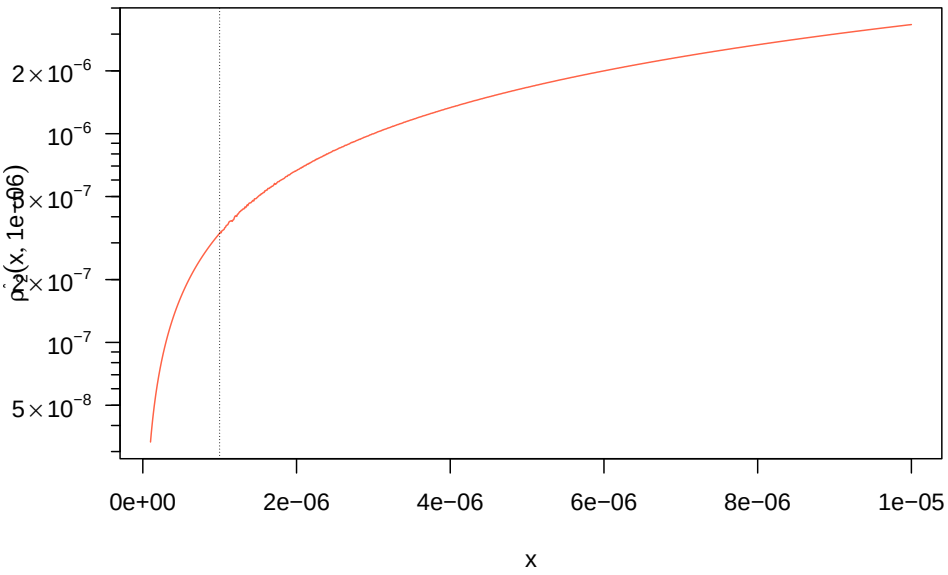
So, `.rhoAmh.2()` using a 4-terms series approximation for $|\theta| <$`e.sml` is much better, but it is still not good enough, as is revealed by drawing it once with its default cutoff `e.sml`$= 10^{-11}$ and then in red with a higher cutoff 10^{-6} (and in log-log and regular y-axis scale):

```
> if(require("sfsmisc")) {
       myAxes <- function(sides) for(s in sides) eaxis(s)
   } else {
       myAxes <- function(sides) for(s in sides)  axis(s)
   }
> rhoAcurve <- function(k, ..., log = "",
                        ylab = substitute({rho^"*"}[2](x,KK), list(KK=k)))
       curve(Vectorize(.rhoAmh.2)(x, k), n=1025, ylab=ylab, log=log,
             xaxt = if(grepl("x", log, fixed=TRUE)) "n" else "s",
             yaxt = if(grepl("y", log, fixed=TRUE)) "n" else "s", ...)
> e.s <- eval(formals(.rhoAmh.2)$e.sml); t0 <- e.s * .99999
> op <- sfsmisc::mult.fig(2, marP = -c(1.4,1,1,1))$old.par
> rhoAcurve(e.s, 1e-18, 1e-1, log = "xy", ylab=""); myAxes(1:2)
> lines(t0, .rhoAmh.2(t0), type="h", lty=3, lwd = 3/4)
> rhoAcurve(1e-6, add=TRUE, col=adjustcolor(2, 1/3), lwd=4)
> rhoAcurve(1e-6, 1e-18, 1, log="x", col="tomato"); myAxes(1)
> par(op)
```



So the default cutoff (10^{-11}) is too small, as the explicit (Nelsen) formula breaks down between the cutoff and $\approx 10^{-7}$. Hence we are aiming for a cutoff $> 10^{-7}$, momentarily $= 10^{-6}$, and zoom into its neighborhood:

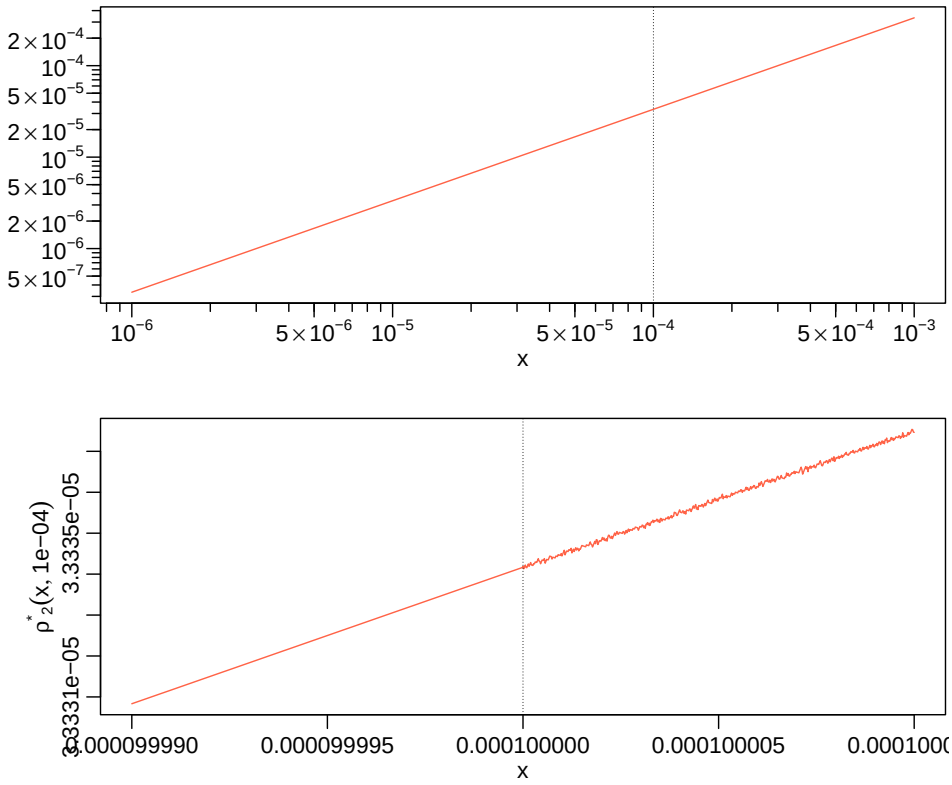
```
> rhoAcurve(1e-6, 1e-7, 1e-5, log = "y", col="tomato"); myAxes(2)
> abline(v=1e-6, lty=3, lwd=1/2)
```



Use still a larger cutoff:

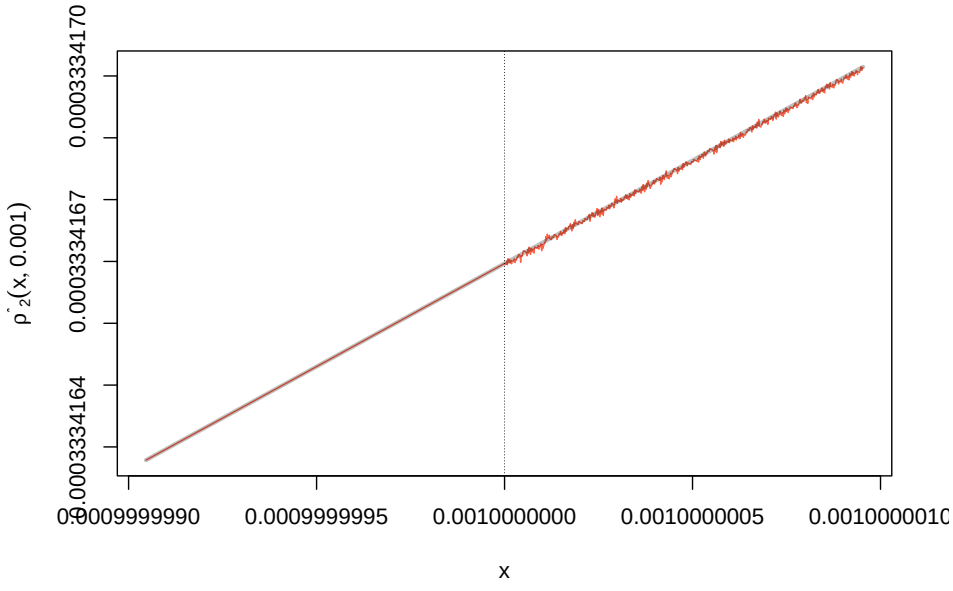
```
> cc <- 1e-4 ; op <- mult.fig(2, marP= -c(1,0,1,1))$old.par
> rhoAcurve(cc, 1e-6, 1e-3, log = "xy", col="tomato",ylab=""); myAxes(1:2)
```

```
> abline(v=cc, lty=3, lwd=1/2)
> ## zoom in extremely:
> rhoAcurve(cc, cc*(1-1e-4), cc*(1+1e-4), col="tomato")
> abline(v=cc, lty=3, lwd=1/2);          par(op)
```



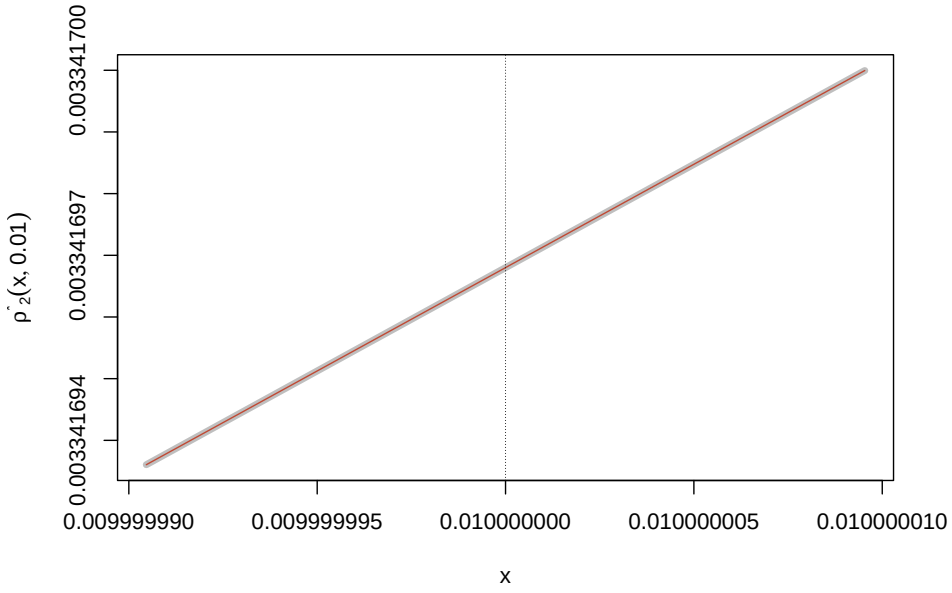
Still larger cutoff:

```
> cc <- 1e-3
> rhoAcurve(cc, cc*(1-2^-20), cc*(1+2^-20), log="y",yaxt="s", col="tomato")
> abline(v=cc, lty=3, lwd=1/2)
> rhoAcurve(cc*10, add=TRUE, col=adjustcolor(1,.25), lwd=3)
```



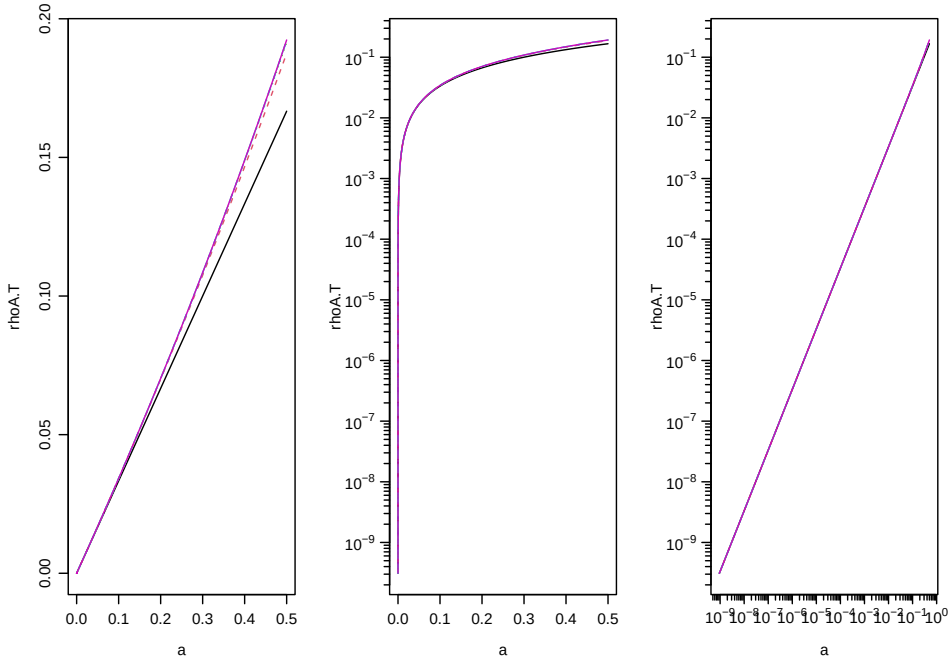
Still larger ...

```
> cc <- 0.01
> rhoAcurve(cc, cc*(1-2^-20), cc*(1+2^-20), log="y",yaxt="s", col="tomato")
> abline(v=cc, lty=3, lwd=1/2)
> rhoAcurve(cc*10, add=TRUE, col=adjustcolor(1,.25),lwd=5)
```



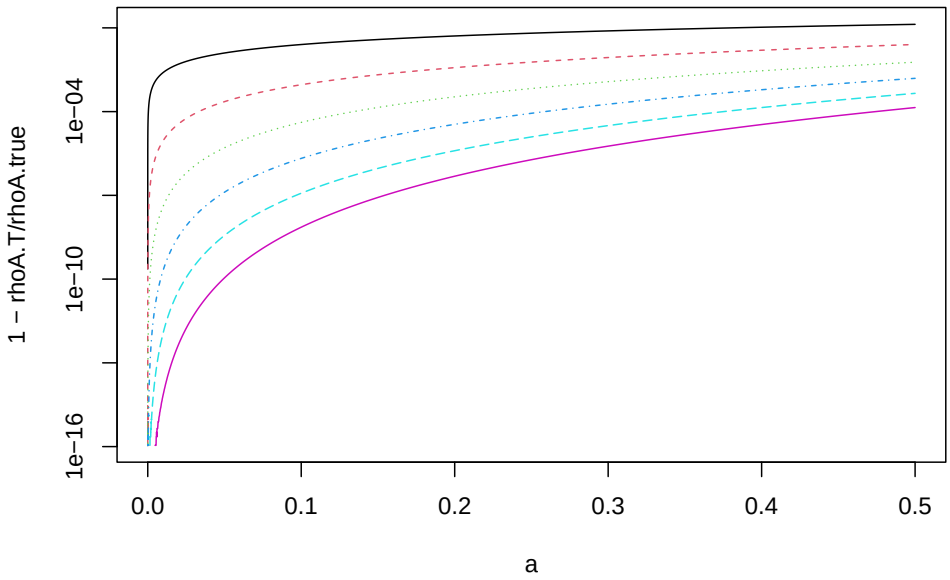
And "visibly", it still seems perfect. This would suggest that a 4-terms approximation is to be preferred to the direct formula for $|\theta < 10^{-3}|$, possibly even $|\theta < 10^{-2}|$. We will determine the best k-terms series approximation for different cutoffs for $k = 1, 2, 3, 4, 5$, in the following. Looking at the series approximations (first order up to 6-th order) a first time,

```
> a <- 2^seq(-30,-1, by = 1/32)# 0 < a <= 0.5
> rhoA.T <-  vapply(1:6, rhoAmh.T, a=a, numeric(length(a)))
> op <- mult.fig(mfcol=c(1,3), mgp=c(2.5,.8,0))$old.par
> matplot(a, rhoA.T, type="l")
> matplot(a, rhoA.T, type="l", log="y", yaxt="n")   ; myAxes(2)
> matplot(a, rhoA.T, type="l", log="xy", axes=FALSE); myAxes(1:2);box()
> par(op)
```



Now, rather look at the *relative* approximation error of the different Taylor series approximations:

```
> rhoA.true <- rhoAmh.T(a,50)
> chk.w.mpfr <- FALSE ## Sys.info()[["user"]] == "maechler"
> if(chk.w.mpfr) {
      require(Rmpfr)## get the "really" "true" values:
      print(system.time(rhA.mp  <- rhoAmh.T(mpfr(a, prec=256), 50))) ## 3.95 sec (lynne)
      print(system.time(rhA.mp1 <- rhoAmh.T(mpfr(a, prec=256), 60))) ## 4.54 sec
      stopifnot(all.equal(rhA.mp, rhoA.true, tol = 1e-15))
          print(all.equal(rhA.mp, rhoA.true, tol = 1e-20)) ## 6.99415....e-17 [64bit, lynne]
      ## see if the 50 terms have converged:
      print( all.equal(rhA.mp, rhA.mp1, tol = 1e-30) )
      ## "Mean relative difference: 2.4958....e-22"
      ## ==> 50 terms seem way enough for double prec
   }
> matplot(a, 1 - rhoA.T / rhoA.true, type="l", log="y")
```



We rather provide a function for *visualizing* the relative approximation errors of the different Taylor series approximations in a flexible way:

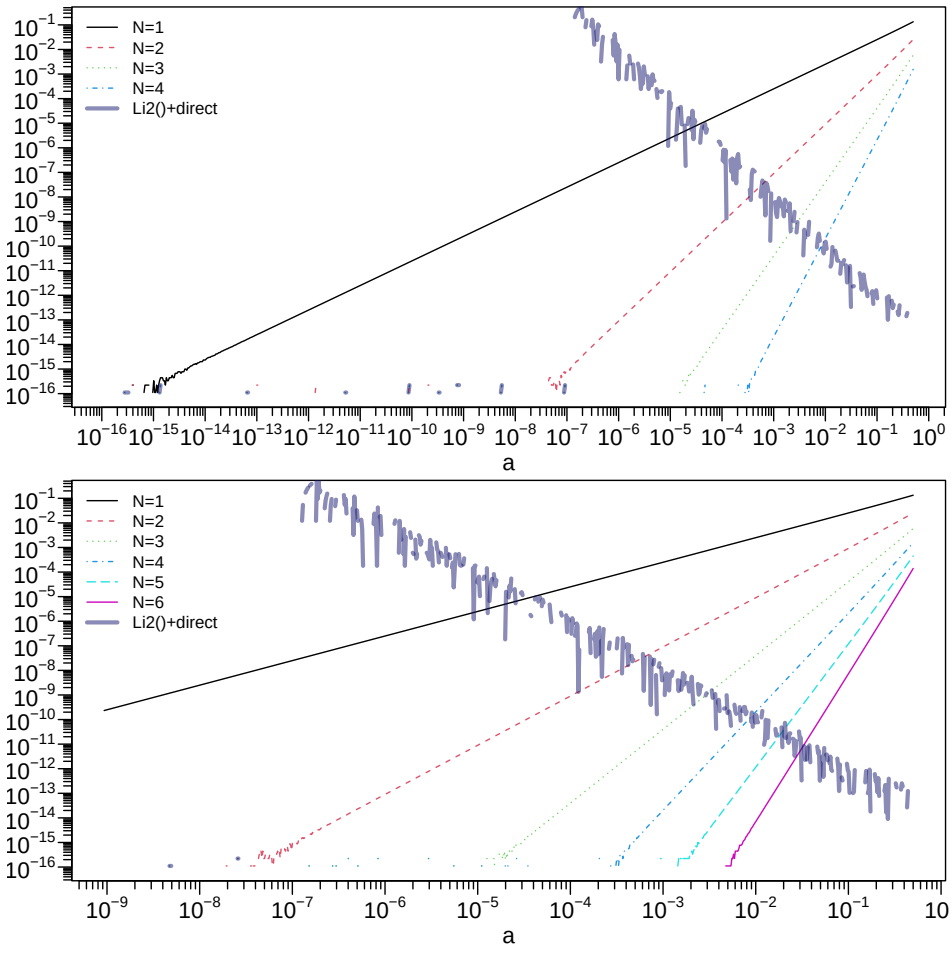
```
> pl.relE.rhoAMH <- function(N.max, N.inf = 50, N.min = 1, l2a.lim = c(-30, -1),
                             n.p.u = 2^round(log2(1000 / diff(l2a.lim))),
                             cut.rA2 = 1e-7,
                             colX = adjustcolor("midnightblue", 0.5), ...)
  {
      stopifnot(length(l2a.lim) >= 2, l2a.lim < 0, n.p.u >= 1,
                N.max >= N.min, N.min >= 1, N.inf > N.max + 4,
                (N3 <- c(N.min, N.max, N.inf)) == as.integer(N3))
      a <- 2^seq(l2a.lim[1], l2a.lim[2], by = 1/n.p.u)
      N.s <- N.min:N.max
      rhoA.true <- rhoAmh.T(a, N.inf)
      rhoA.T <- vapply(N.s, rhoAmh.T, a=a, numeric(length(a))) # matrix
      rhoA.v2 <- Vectorize(.rhoAmh.2)(a, cut.rA2) # "Li2()+direct" below

      ## matplot() compatible colors and lty's
      cols <- palette()[1 + (N.s-1) %% 6]
      ltys <- (1:5)    [1 + (N.s-1) %% 5]
      matplot(a, 1 - rhoA.T / rhoA.true, type="l", log="xy",
              col=cols, lty=ltys, axes=FALSE, frame=TRUE, ...)
      myAxes(1:2)
      lines(a, 1 - rhoA.v2 / rhoA.true, col= colX, lwd=3)
      legend("topleft", c(paste0("N=",N.s), "Li2()+direct"),
             col=c(cols, colX), lty=c(ltys, 1), lwd=c(rep(1,length(N.s)), 3),
             cex=.75, bty="n")
      invisible(list(a=a, rhoA.T=rhoA.T, rhoA.v2 = rhoA.v2))
  }
```

Note that the "Li2()+direct" comparison is only for $a = \theta > 10^{-7}$, as that is used as cutoff per default, `cut.rA2 = 1e-7`. And now look at the "very nice" pictures, using `l2a`$= \log_2(a)$ to choose the range of $a = \theta$:

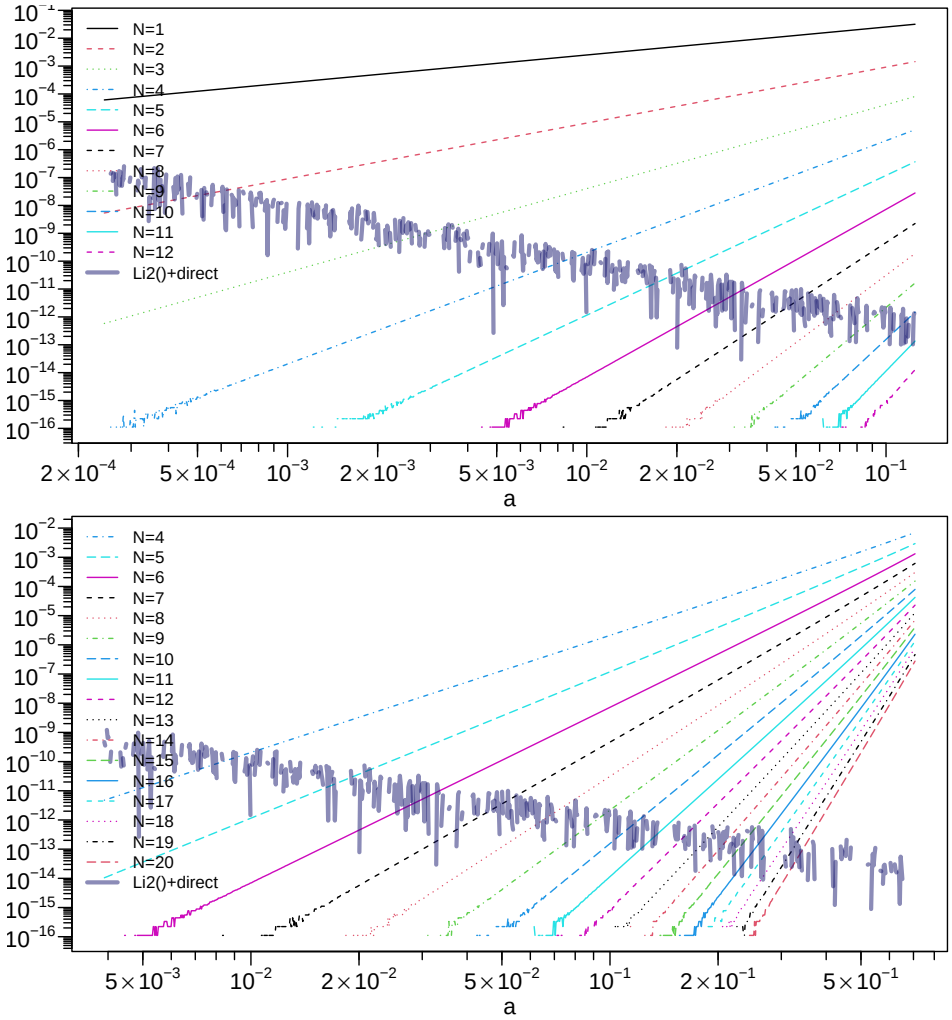
```
> op <- mult.fig(2, marP=-c(1.5,1.5,2,1))$old.par
> pl.relE.rhoAMH(4, l2a=c(-53,-1), ylab="")
```

```
> pl.relE.rhoAMH(6,                    ylab="")
```



Successively zooming in "to the right", to larger a, first, with range $2^{-12} - 2^{-3}$, and up to 12 terms, then zooming into range $2^{-8} - 2^{-.5}$, and using 20,

```
> mult.fig(2, marP=-c(1.5,1.5,2,1))
> pl.relE.rhoAMH(12, l2a=c(-12, -3),             ylab="")
> pl.relE.rhoAMH(20, l2a=c(-8, -.5), N.min = 4, ylab="")
```

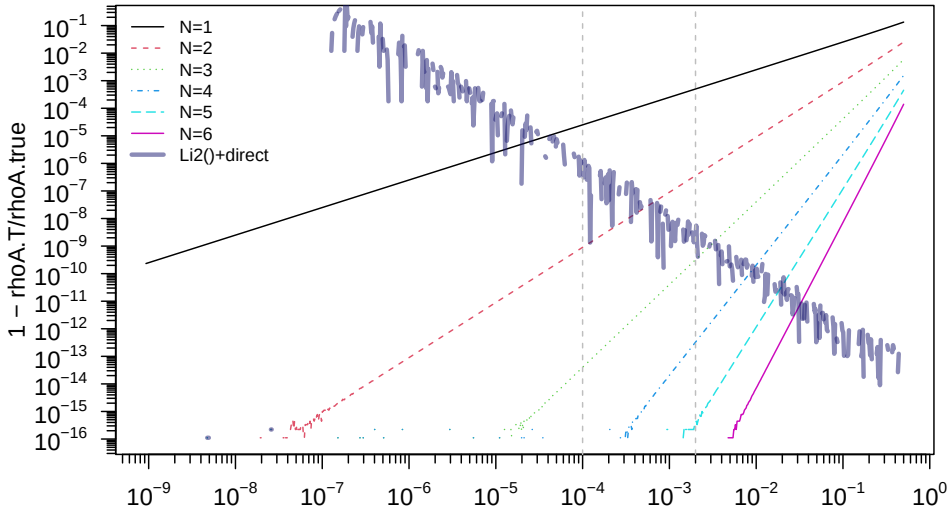


The next one is "just for fun", to see if there is consistency when $N \to N_\infty$, i.e., our `N.inf = 50`, and not shown here:

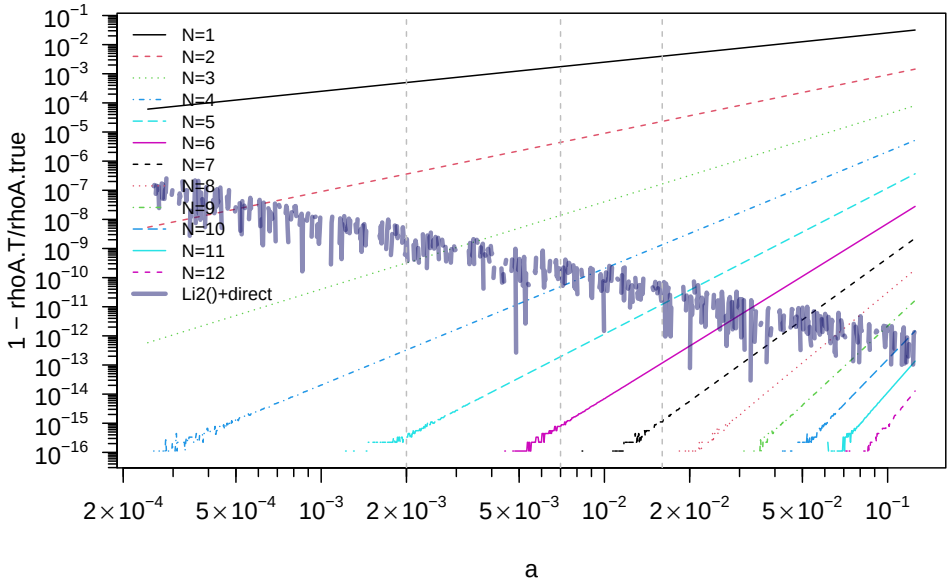
```
> par(op); pl.relE.rhoAMH(40, l2a=c(-5, -.5), N.min = 10)
```

The following plots are now used to read off the final cutoff used for the (hidden) `.rhoAmhCopula()` function in package **copula** which underlies `rho(amhCopula(.))`:

```
> pl.relE.rhoAMH(6)
> abline(v=1e-4, col="gray", lty=2)#-> N=2 cutoff
> abline(v=2e-3, col="gray", lty=2)#-> N=3 cutoff
```




```
> pl.relE.rhoAMH(12, l2a=c(-12, -3))
> abline(v= 2e-3, col="gray", lty=2)#-> N=3 cutoff
> abline(v= 7e-3, col="gray", lty=2)#-> N=4 cutoff
> abline(v=16e-3, col="gray", lty=2)#-> N=5 cutoff
```



Consequently, the implementation in **copula** is

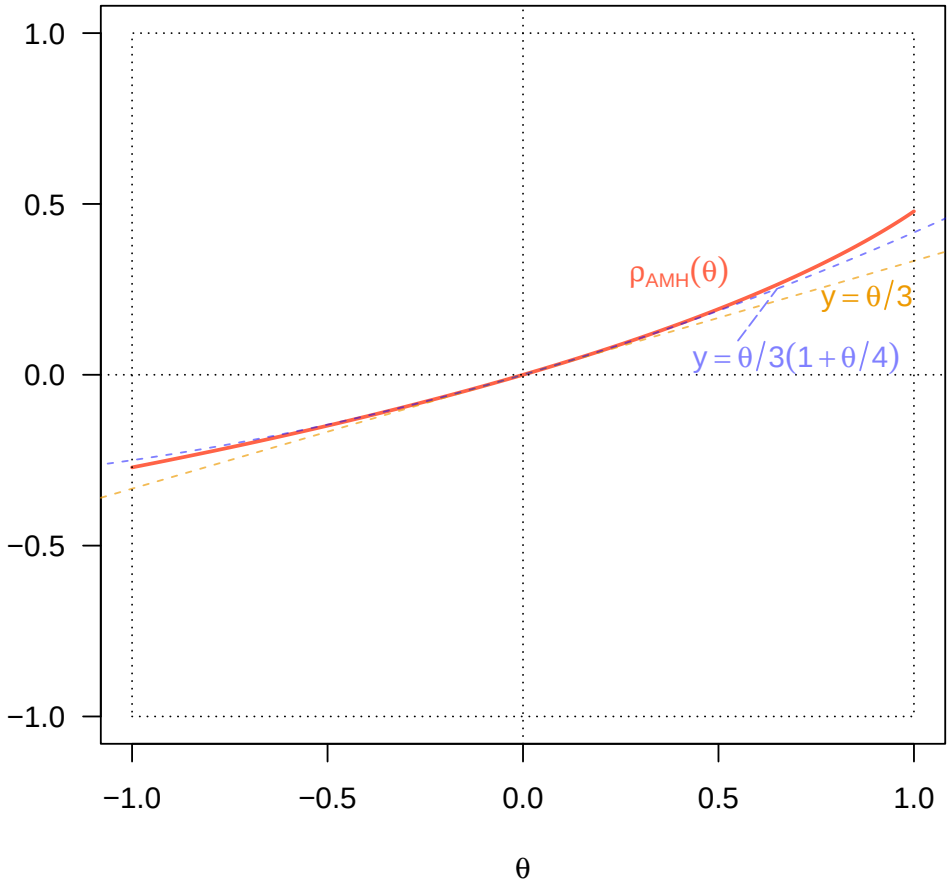
```
> copula ::: .rhoAmhCopula

function (a)
{
    if (is.na(a))
        return(a)
    aa <- abs(a)
    if (aa < 7e-16)
        a/3
    else if (aa < 1e-04)
        a/3 * (1 + a/4)
```

```
    else if (aa < 0.002)
        a * (1/3 + a * (1/12 + a * 3/100))
    else if (aa < 0.007)
        a * (1/3 + a * (1/12 + a * (3/100 + a/75)))
    else if (aa < 0.016)
        a * (1/3 + a * (1/12 + a * (3/100 + a * (1/75 + a/147))))
    else {
        3/a * (4 * (1 + 1/a) * dilog(a) - (if (a < 1)
            8 * (1/a - 1) * log1p(-a)
        else 0) - (a + 12))
    }
}
```

visualized on its full range $[-1, 1]$,

```
> rhoAMH <- Vectorize(copula:::.rhoAmhCopula)
> curve(rhoAMH, n=1025, -1, 1, ylim= c(-1,1), xlab = quote(theta),
         ylab="", col="tomato", lwd=2, las=1)
> abline(0, 1/3, lty=2, col=(adjustcolor(c2 <- "orange2", 2/3)))
> curve(x/3*(1+x/4), lty=2, col=(adjustcolor(c3 <- "blue", 1/2)),
         -1.1,1.1, add=TRUE); x. <- .65
> text(.4 , .3 , quote(rho[plain(AMH)](theta)),col="tomato")
> text(.88, .23, quote(y == theta/3), col=c2)
> text(.7,  .05, quote(y == theta/3*(1+theta/4)), col=adjustcolor(c3, 1/2))
> segments(.55, .10, x., x./3*(1+x./4), lty="82", col=adjustcolor(c3, 1/2))
> abline(h=0,v=0, lty=3); rect(-1,-1,1,1, lty=3)
```



Finally, we may add some simple tests, that the **copula** package's `rho(<amhCopula>, *)` did not fulfill because of the notorious cancellations, previously. Note that in fact, we are only looking at very small (positive) θ, and checking that already the *first* two order series approximations,

$$\rho_{\mathrm{AMH}}(\theta) \approx \frac{\theta}{3}(1 + \frac{\theta}{4}) \approx \theta/3 \tag{13}$$

are all already good approximations or very accurate, depending on $|\theta|$:

```
> t0 <- seq(-1,1, by=2^-8)[1:512]
> t1 <- seq(-1/2, 1/2, by = 2^-8)
> th <- 10^-(6:99); i <- -(1:9)
> rth <- rhoAMH(th)
> stopifnot(all.equal(rhoAMH(1), 4*pi^2 - 39, tol = 8e-15),# <- gave NaN
            all.equal(rhoAMH(t0), t0/3 * (1 + t0/4), tol = 0.06),
            all.equal(rhoAMH(t1), t1/3 * (1 + t1/4), tol = 1/85),
            all.equal(rth,      th / 3 * (1 + th/4), tol = 1e-15),
            all.equal(rth,      th / 3, tol = 1e-6),
            all.equal(rth[i], th[i]/ 3, tol = 6e-16))
> th <- 10^-(16:307)
> stopifnot(all.equal(th/3, rhoAMH(th), tol=4e-16),
            rho(amhCopula(0, use.indepC="FALSE")) == 0)
```

Session Information

```
> toLatex(sessionInfo(), locale=FALSE)
```

- R version 4.3.3 Patched (2024-02-29 r86017), `x86_64-pc-linux-gnu`
- Running under: `Debian GNU/Linux 12 (bookworm)`
- Matrix products: default
- BLAS: `/srv/R/R-patched/build.24-03-02/lib/libRblas.so`
- LAPACK: `/srv/R/R-patched/build.24-03-02/lib/libRlapack.so` ; LAPACK version3.11.0
- Base packages: base, datasets, grDevices, graphics, grid, methods, parallel, splines, stats, stats4, tools, utils
- Other packages: Rmpfr 0.9-5, VGAM 1.1-10, abind 1.4-7, bbmle 1.0.25.1, copula 1.1-4, gmp 0.7-4, gridExtra 2.3, gsl 2.1-8, lattice 0.22-5, mev 1.16, qrng 0.0-10, randtoolbox 2.0.5, rngWELL 0.10-10, rugarch 1.5-1, sfsmisc 1.1-17
- Loaded via a namespace (and not attached): ADGofTest 0.3, DistributionUtils 0.6-1, GeneralizedHyperbolic 0.8-6, KernSmooth 2.23-22, MASS 7.3-60.0.1, Matrix 1.7-0, R6 2.5.1, Rcpp 1.0.12, Rdpack 2.6, Rsolnp 1.16, Runuran 0.38, SkewHyperbolic 0.4-2, alabama 2023.1.0, bdsmatrix 1.3-7, bslib 0.6.1, cachem 1.0.8, cli 3.6.2, compiler 4.3.3, digest 0.6.34, evaluate 0.23, fastmap 1.1.1, glue 1.7.0, gtable 0.3.4, highr 0.10, htmltools 0.5.7, jquerylib 0.1.4, jsonlite 1.8.8, knitr 1.45, ks 1.14.2, lifecycle 1.0.4,

mathjaxr 1.6-0, mclust 6.1, mvtnorm 1.2-4, nleqslv 3.3.5, nloptr 2.0.3, numDeriv 2022.9-1, partitions 1.10-7, pcaPP 2.0-4, polynom 1.4-1, pracma 2.4.4, pspline 1.0-19, rbibutils 2.2.16, rlang 1.1.3, rmarkdown 2.25, sass 0.4.8, spd 2.0-1, stabledist 0.7-2, truncnorm 1.0-9, xfun 0.42, xts 0.13.2, yaml 2.3.8, zoo 1.8-13

```
> unlist(packageDescription("copula")[c("Package", "Version", "Date")])

     Package      Version         Date
   "copula"      "1.1-4" "2024-02-07"
```

References

Hofert M, Maechler M (2011). "Nested Archimedean Copulas Meet R: The nacopula Package." *Journal of Statistical Software*, **39**(9), 1–20. ISSN 1548-7660. URL http://www.jstatsoft.org/v39/i09.

Joe H (1997). *Multivariate Models and Dependence Concepts*. Chapman & Hall/CRC.

Nelsen RB (2007). *An Introduction to Copulas*. 2nd edition. Springer-Verlag, New York.

Sklar A (1959). "Fonctions de Répartition à n Dimensions et Leurs Marges." *Publications de L'Institut de Statistique de L'Université de Paris*, **8**, 229–231.

Affiliation:

Martin Mächler
Seminar für Statistik, HG G 16
ETH Zurich
8092 Zurich, Switzerland
E-mail: maechler@stat.math.ethz.ch
URL: http://stat.ethz.ch/people/maechler